About Complementarity

Sobre a Complementaridade

Michael Friedrich Otte[a]*; Luiz Gonzaga Xavier de Barros[b]

[a]Universidade Federal de Mato Grosso. MT, Brasil.
[b]Universidade Anhanguera de São Paulo, Programa de Pós-Graduação *Stricto Sensu* em Educação Matemática. SP, Brasil.
*E-mail: michaelontra@aol.com
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Abstract

Niels Bohr, around 1930, when he tried to describe some processes in quantum physics, created the term 'complementarity'. Bohr believed in the general epistemological and metaphysical significance of this principle. Since then several authors have used this term to capture the essential aspects of the cognitive and epistemological development of scientific and mathematical concepts (Otte and Steinbring, 1977; Kuyk, 1977; Otte, Keitel and Seeger, 1980; Otte, 1984, 1990, 1994; Douady, 1991; Sfard, 1991; Jahnke, 1992). However, what is complementarity? It is an idea or a concept! Being an idea, complementarity finds different expressions within different contexts. Mathematicians do not know what to do with ideas and they want definitions instead. In this article, we discuss different aspects of complementarity trying to clear the matter.
Keywords: Mathematics Education. Philosophy of Mathematics. Complementarity.

Resumo

Niels Bohr, em torno de 1930, quando tentava descrever alguns processos da física quântica, criou o termo "complementaridade". Bohr acreditava na significância metafísica e epistemológica desse princípio. Desde então diversos autores têm utilizado esse termo para capturar os aspectos essenciais do desenvolvimento epistemológico e cognitivo de conceitos matemáticos e científicos (Otte and Steinbring, 1977; Kuyk, 1977; Otte, Keitel and Seeger, 1980; Otte, 1984, 1990, 1994; Douady, 1991; Sfard, 1991; Jahnke, 1992). Todavia, o que é a complementaridade? É uma ideia ou um conceito! Sendo uma ideia, a complementaridade revela diferentes expressões em diferentes contextos. Os matemáticos não sabem o que fazer com ideias e querem, em vez disso, definições. Neste artigo, discutimos diferentes aspectos da complementaridade tentando esclarecer o assunto.
Palavras-chaves: Educação Matemática. Filosofia da Matemática. Complementaridade.

1 Introduction

When being asked, how this concept came about, one has to answer in historical terms. We find the first philosophical expression of the idea of complementarity in Kant's epistemology and in his assertion that all human knowledge "springs from two fundamental sources of the mind; the first is the capacity of receiving representations …, the second is the power of knowing an object through these representations …. Through the first, an object *is given* to us, through the second the object is *thought* in relation to that [given] representation. … Intuition and concepts constitute, therefore, the elements of all our knowledge" (Kant, 1787, p.74).

Kant's philosophy is, as we know, the result of a thorough rumination and distillation of Newton's scientific achievements and his philosophy expresses the intellectual spirit of these achievements. Kant wanted to unite necessity and objectivity of knowledge and took Newton`s science as the starting point of an epistemological analysis. For example, Newton's observations about the origin of objective precision of mathematical natural philosophy (see preface of the first edition of his *Mathematical Principles of Natural Philosophy*) led Kant to understand that the source of knowledge is to be found in the constructive activity of the subject, implying in particular that this activity is objectively constrained. Constrained, not by what we think, or believe, but by the structures of space and time, by the laws that govern, let`s say, the possibilities of an engineer of symbolic constructions, for example.

Newton (1729, p. xvii) had written in the preface:

> The ancients considered mechanics in a twofold respect; as rational, which proceeds accurately by demonstration, and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. However, as artificers do not work with perfect accuracy, it happens that mechanics is so distinguished from geometry that what is perfectly accurate is called geometrical; what is less so, is called mechanical. However, the errors are not in the art, but in the artificers.

Now the perfect artificer is God, and in our thought experiments we can imagine, or imitate his perfection by acting on algebraic or geometrical diagrams. Such a diagram is established by the complementarity of icons and indices. And although iconicity represents the dominant character of mathematical diagrams, it is indexicality, what in particular makes the semiotic approach unavoidable, because it shows that mathematical reasoning is contextual like all other reasoning. The relevant contexts are semiotical contexts.

2 The Intension-Extension Complementarity

In fact, it is sometimes claimed that it is with the notion of index "that Peirce is at once novel and fruitful" (Sebeok, 1995, p. 15). Peirce saw, Sebeok continues, "as no one before him had, that indication is a mode of signification as indispensable as it is irreducible". So, we come back to the intensions and extensions of symbolic representations as the basis of the idea of complementarity.

In Otte (2003, p. 204) the following description of the notion of complementarity is giving by saying that by the occasion of a lecture given by H. Wiener in 1891, Hilbert made a remark, which contains the axiomatic standpoint in a nutshell: "It must be possible to replace in all geometric statements the words point, line, plane by table, chair, mug". Hilbert's remark is usually interpreted as expressing the tendency towards a de-ontologization of modern axiomatized mathematics. This is not so. To the contrary, mathematics distinguishes itself from logic exactly by the fact that it has objects. As implied by Gödel's incompleteness theorem, all axiomatic descriptions must necessarily remain incomplete.

Any formal theory has various intended applications or non-isomorphic models, and what the axioms describe are concepts or classes of objects, rather than particular objects themselves. In this respect, mathematical axioms resemble natural laws. And like the latter, they have to be supplemented by an indication of the domain of objects to which they apply. A mathematical theory should this be conceived of as a pair consisting of a system of axioms, that is, a syntactic structure together with a set of models or intended applications.

Modern axiomatized theories become, on the one hand, intensional theories in the sense that the axioms as a set of postulates not only determine the intensions of the theoretical terms, but also constitute the extensions or referents. In Euclidean geometry, the objects about which the theory speaks seem to be given by intuition, and independently of the theory. In Hilbert's geometry, the situation is quite different, as the above quotation shows. To answer questions such as, what is a point? or, what is a number? one provides the respective axiomatic descriptions of the relations or laws, which govern these entities. On the other hand, as implied by Gödel's incompleteness theorem, all axiomatic descriptions must necessarily remain incomplete. And any formal theory has various intended applications or models.

We shall conclude that mathematical terms, the senses (or intensions) of which are given by the systems of axioms and law-like statements, can be (and are) used 'attributively' as well as 'referentially'. That is, the terms occurring in the axioms of a theory can be regarded, on the one hand, as giving descriptions of their referents, to be applied to those and only those entities with reference to which they are true. On the other hand, the terms contained in the axioms or in mathematical discourse in general can be used 'referentially', too. In this case, we do not regard the expressions of the theory as referring to those objects, which satisfy the given denotation, but as saying something about objects fixed independently of the given description. In group theory, for example, one uses axiomatic presentations of groups together with group representations (in terms of linear transformations or in terms of permutations or whatever).

To illustrate the latter point let us discuss the following example. Suppose an English tourist visiting Amazonia sees a biggish animal near the shore of a lake and asks what kind of animal this is. He is told that what is seen is a *Capivara*. As the tourist cannot speak Brazilian Portuguese, this is only an indexical or referential designation, which leaves him without any representation for the moment. If he is offered, to relieve his frown, an anglicization in the form of 'water hog', his face lights up and he says 'aha', actually believing to have understood what it is, the fact being that he is able to link something meaningful with the words of 'water' and 'hog'. This is thus a case of some kind of descriptive designation, which has the disadvantage, however, of creating false notions. For the *Capivara* is no swine at all, but a grass-eating rodent. The Amazonian is in the opposite situation, as for him the Indian name of *Capivara* has the meaning of 'grass-eater', while the designation 'water hog' tells him absolutely nothing.

Now such a referential use sometimes serves as the starting point of further observations if a motive or curiosity results. After some time, the tourist may observe some characteristics and habits of the *Capivara*, and then will be able to say, "*Capivaras* are good swimmers and divers", or "the *Capivara* lives in family groups", etc. Gradually, the use of the term changes and it is transformed into a description. And indeed, theories *in statu nascendi* are mainly used 'referentially' by their exponents as well as by their opponents, while having reached their zenith, they are used 'attributively', until a new theory emerges and ascends to its zenith, when the former theory is used 'referentially' again.

The interdependence of attributive *versus* referential uses of terms is much more prominent with respect to mathematical concepts than in empirical ones, because mathematical objects firstly do not exist independently of any representation and, secondly because their instrumental character is much more pronounced. In pure number theory, numbers and number relations are the objects of study, in most number-theoretic propositions numbers occur as nouns, whereas in applied mathematics number terms are used predicatively or as adjectives. Numbers seem to have come into being as adjectives.

3 Periods of History of Mathematics

Following Pierre Boutroux (1920), in that same paper, the history of mathematics since Antiquity is considered as divided into essentially three periods, which could roughly be indicated by the names:

Period 1: Plato/Euclid

Period 2: Descartes/Newton

Period 3: Bolzano/Cantor and Hilbert.

Boutroux sees the essential revolution or break to occur between Period 2 and Period 3, while classifying Period 1 and Period 2 as both dedicated to a synthetical ideal of mathematics, which is characterized by a pre-established harmony "entre le but et la méthode de la science mathématique, entre les objets que poursuit cette science et les procédés qui lui permettent d'atteindre ces objets" (Boutroux, 1920, p.193).

But, as was said already, we have a change of the dominant rationality type occurring already during the Scientific Revolution of the 16th/17th centuries.

Plato's geometrical cosmology had been designed in a bold and almost romantic spirit, no doubt about that. And what about Galilei and his claim that the Great Book of Nature is written in terms of geometrical figures? This rings definitely like Platonism, doesn't!

Koyré (1943, p.400) writes:

> The name of Galileo Galilei is indissolubly linked with the scientific revolution of the sixteenth century, one of the profoundest, if not the most profound, revolution of human thought since the invention of the Cosmos by Greek thought: a revolution which implies a radical intellectual 'mutation', of which modern physical science is at once the expression and the fruit. This revolution is sometimes characterized, and at the same time explained, as a kind of spiritual upheaval, an utter transformation of the whole fundamental attitude of the human mind; the active life, the vita active, taking the place of the *vita contemplativa*, which until then had been considered its highest form. Modern man seeks the domination of nature, whereas medieval or ancient man attempted above all its contemplation.

Koyré (1943, p. 400) does not think that this picture is completely correct. He says: "Galileo did not learn his business from people who toiled in the arsenals and shipyards of Venice. Quite the contrary: he taught them theirs". And it is still true that experimentation formed an essential part of "modern science".

Koyré (1943, p. 400) again:

> It is not experience, but experiment, which played - but only later - a great positive role. Experimentation is the methodical interrogation of nature, an interrogation, which presupposes and implies a language in which to formulate the questions, and a dictionary, which enables us to read and to interpret the answers. For Galileo, as we know well, it was in curves and circles and triangles, in mathematical or even more precisely, in geometrical language - not in the language of common sense or in that of pure symbols - that we must speak to Nature and receive her answers. Yet obviously, the choice of the language, the decision to employ it, could not be determined by the experience, which its use was to make possible. It had to come from other sources.

These words by Koyré could certainly be most appropriately interpreted, by saying that the opposition between mathematical speculation and cosmic order, which had dominated Plato's philosophy, has been transformed and changed by projecting it onto a new reality, namely, reality, understood as objective human activity and practice. Kant's emphasis on the importance of epistemology and his assertion that all human knowledge "springs from two main sources in the mind", is an expression of these changes. The science and philosophy of Greek antiquity had been riddled by the dichotomy of the discrete and the continuous. Zenon's paradoxes of motion are an expression of that fact.

These historical considerations and our reflections about the various styles of thinking dominating them brings the idea that the notion of complementarity could be somewhat to similar to what philosophers have called dialectics. So let us consider some paragraphs of Engels' book "*Dialectics of Nature*", which are about the very same historical developments.

Engels (1907, p. 15) writes:

> The philosophy of antiquity was primitive, spontaneously evolved *materialism*. As such, it was incapable of clearing up the relation between mind and matter. But the need to get clarity on this question led to the doctrine of a soul separable from the body, then to the assertion of the immortality of this soul, and finally to monotheism. The old materialism was therefore negated by *idealism*. But in the course of the further development of philosophy, idealism, too, became untenable and was negated by modern materialism. This modern materialism, the negation of the negation, is not the mere re-establishment of the old, but adds to the permanent foundations of this old materialism the whole thought-content of two thousand years of development of philosophy and natural science, as well as of the history of these two thousand years. It is no longer a philosophy at all, but simply a world outlook, which has to establish its validity and be applied not in a science of sciences standing apart, but in the real sciences. Philosophy is therefore 'sibilated' here, that is, 'both overcome and preserved'; overcome as regards its form, and preserved as regards its real content' (Chapter XIII.).

From *Dialectics of Nature***:** "But what especially characterizes this period is the elaboration of a peculiar general outlook, in which the central point is the view of the absolute immutability of nature. In whatever way nature itself might have come into being, once present it remained as it was as long as it continued to exist. The planets and their satellites, once set in motion by the mysterious "first impulse", circled on and on in their predestined ellipses for all eternity, or at any rate until the end of all things. The stars remained forever fixed and immovable in their places, keeping one another therein by universal gravitation.

The earth had persisted without alteration from all eternity, or, alternatively, from the first day of its creation. The "five continents" of the present day had always existed, and they had always had the same mountains, valleys, and rivers, the same climate, and the same flora and fauna, except in so far as change or cultivation had taken place at the hand of man. The species of plants and animals had been established once for all when they came into existence; like continually produced like, and it was already a good deal for Linnaeus to have conceded that possibly here and there new species could have arisen by crossing. In contrast to the history of humanity, which develops in time, there was ascribed to the history of nature only an unfolding in space. All change, all development in nature, was denied. Natural science, so revolutionary at the outset, suddenly found itself confronted by an out-and-out conservative nature in which even to-day everything was as it had been at the beginning and in which - to the end of the world or for all eternity - everything would remain as it had been since the beginning.

High as the natural science of the first half of the eighteenth century stood above Greek antiquity in knowledge and even in the sifting of its material, it stood just as deeply below Greek antiquity in the theoretical mastery of this material, in the general outlook on nature. For the Greek philosophers the world was essentially something that had emerged from chaos, something that had developed, that had come into being. For the natural scientists of the period that we are dealing with it was something ossified, something immutable, and for most of them something that had been created at one stroke.

Science was still deeply enmeshed in theology. Everywhere it sought and found its ultimate resort in an impulse from outside that was not to be explained from nature itself. Even if attraction, by Newton pompously baptized as "universal gravitation", was conceived as an essential property of matter, whence comes the unexplained tangential force, which first gives rise to the orbits of the planets? How did the innumerable varieties of animals and plants arise? And how, above all, did man arise, since after all it was certain that he was not present from all eternity? To such questions natural science very frequently are answered by making the creator of all things responsible. Copernicus, at the beginning of the period, writes a letter renouncing theology; Newton closes the period with the postulate of a divine first impulse. The highest general idea to which this natural science attained was that of the purposiveness of the arrangements of nature, the shallow teleology of Wolff, according to which cats were created to eat mice, mice to be eaten by cats, and the whole of nature to testify to the wisdom of the creator".

Therefore, during the $19^{th}/20^{th}$ centuries the relationship between mathematics and science, on the one side, and philosophy, on the other side, had to change and to make progress. Therefrom the new interest in Plato resulted, such that Whitehead could say "The safest general characterization of the European philosophical tradition is that it consists of a series of footnotes to Plato" (Whitehead, 1929, p.15).

Mathematics is essentially a science of the identical and the different, or about equality and difference. Thus, it is essentially about *Number*. Therefore, mathematics should be arithmetized to begin with. All mathematics, says Russell, for example, "including analytical geometry, may be considered as consisting wholly of propositions about natural numbers" (Russell, 1967, 51).

Moreover, if the world should be mathematized it must be finished, static and discrete. Thence Pythagoras dream of arithmetizing it. Splenger, in expressed these desires:

> In number, as the sign of the completed and limited, is therefore contained the nature of all reality, that has become recognized and limited at the same time, as Pythagoras or who it was otherwise, realized with innermost certainty as a result of a great, quite religious intuition (Splenger, 1918, p. 110).

One of the most important discoveries of the Pythagorean School is without doubt that of the incommensurability of the side and the diagonal of the square. And this implied the failure of arithmetization and was a rejection of the Pythagorean view that the realm of number provides a model or an image of the world. This failure to establish number, as the basic element of the Universe was disastrous.

However, it seems that Plato had changed the program replacing arithmetic by geometry as the realm, which provides the basic elements of real ontology. If the square root of two or three cannot be rendered rationally in terms of number, than one might accept their geometric representations and try to understand the building blocks of the universe in these terms. Karl Popper has in particular contributed to such a view.

He writes:

> The discovery of the irrationality of the square root of two destroyed the Pythagorean program of arithmetizing geometry and with it, it appears the vitality of the Pythagorean order itself. … It appears that the breakdown of the Pythagorean program, i.e. of the arithmetical method of geometry, led to the development of the axiomatic method of Euclid, that is to say of a new method, which has, on the one side, to rescue, what can be rescued (including the method of rational proof) and, on the other side, accept the irreducibility of geometry to arithmetic (Popper, 1945, p. 15).

Since the 17^{th} century, the *aporias* of ancient thought have gradually been replaced by complementarities. Facts and theories become complementary elements of the system of human activity.

And think of the *function* concept, which was decisive, as much in the development of the new calculus, as to classical mechanics. While Galileo was primarily responsible for dismantling the Platonic - Ptolemaic cosmology, the creation of the corpuscular-mechanistic worldview was the achievement of a number of investigators, owing its final culmination to Newton. In this sense, it is right when claiming that the Scientific Revolution essentially consisted in raising mechanics to the level of philosophy. Newton's approach to calculus rests firmly on the conception of continua as being generated by motion. And motion is mathematically modelled in terms of the mathematical function concept.

To understand a mathematical function means to understand the complementarity of formula and relation, of algorithm and natural law. In the mathematics of the 17th/18th centuries, discontinuous functions could not be represented, because a function was an analytical law. A geometrical curve, on the other hand, was called continuous if it could be represented by an analytical function (Euler, 1748). But this characterization proved to be incoherent.

Cauchy, after having demonstrated the inconsistency of these efforts (Grattan-Guinness, 1970), revised the whole approach of the principle of continuity, transforming mathematics into extensional theory. A function in Cauchy's or Dirichlet's sense can be seen as an equivalence class of analytic expressions or formulae, where the equivalence relation is based on the axiom of extensionality. This switch from an intensional to an extensional view has made it possible since Cauchy to single out sets of functions by certain of their properties, and in general to reason about them, without representing them explicitly (Cauchy, 1821, p.99-100).

This kind of conceptual reasoning characterizes, according to Boutroux, the third period in the evolution of mathematics, the period of *complementarity* proper, which begins around 1800. At the beginning of the 19^{th} century, pure mathematics emerges, based on proof analysis and of the creation of ever more abstract concepts, and the harmony between means and objects of mathematical activity begins to collapse. Pure mathematics is the child of an explosive growth of mathematical activity that may be briefly characterized by stating that a great number of connections between apparently very different results and problems were detected, for the first time in the history of mathematics. Descartes' discovery of analytical geometry already initiated a process that indeed became dominant from the early nineteenth century only.

4 Zeno's Paradox

The classical example that clearly demonstrates the complementarity of the algorithmic and the relational, or of the discrete and the continuous in the function concept derives from Zeno's paradox of Achilles and the tortoise. This paradox has become so prominent because it seems to be a striking expression of the classical Platonic rationality type and its difficulties.

One can find this paradox in any school textbook of mathematics. It is usually paraphrased as follows:

Suppose Achilles runs ten times as fast as the tortoise and gives him a hundred yards start. To win the race Achilles must first make up for his initial handicap by running a hundred yards; but when he has done this and has reached the point where the tortoise started, the animal has had time to advance ten yards. While Achilles runs these ten yards, the tortoise 1gets one yard ahead; when Achilles has run this yard, the tortoise is a tenth of a yard ahead; and so on, without end. Achilles never catches the tortoise, because the tortoise always holds a lead, however small. Zeno's problem is a paradox of movement. In physics, movements are understood as continuous functions of time in three-dimensional space g(t) = (x(t), y(t), z(t)), with t as a time parameter: "We talk of a movement when the (space) coordinates of the body change over the course of time", states a randomly selected physics textbook.

And the continuous function, as a model of movement, actually very clearly reflects the double character of this concept: On the one hand, it contains discrete aspects, such as it permits me to calculate single values, when it is written as a formula. On the other hand, it emphasizes continuous aspects, for example, in the illustration of the functional graph that gives a qualitative overall idea of the function (= movement). The function is simultaneously both qualitative and quantitative, knowledge (overall idea) and tool (calculation formula) in one.

Zeno wanted to know the movement perceived and "measured" at given positions, at points, whose distances converge toward zero.

$$x_0 = 0 \quad \text{and} \quad x_{n+1} = (x_n/10) + 100 \text{ yard} \quad \text{for } n = 0, 1 \ldots.$$

He disguises this procedure for "measuring" movement in the above story: When Achilles is at x_n, the tortoise is already at $x_{n+1} > x_n$! Here he exposes his audience to the fallacy that places the one-sided discrete view of movement, in contrast to knowledge about the continuous course together with the knowledge that the slower one must finally be overtaken by the faster despite such a large start. If we accept this exclusively discrete approach, we agree that Achilles must first reach all the points that the tortoise has already reached (by which the tortoise is naturally always a little bit further in front!). Therefore, what we are actually saying is that Achilles can reach only these points, that these are quasi the only positions that he can reach, or at least, the only ones that determine his position. We quasi encapsulate Achilles' movement within that of the tortoise; we chain it to the latter.

We have to symmetrize our perspective by adopting a relational point of view. Precisely speaking, the task is as follows: Achilles runs ten times as fast as the tortoise, though the tortoise has one hundred yard start. For each of the stages, x (x > 0), covered by Achilles, the tortoise has crawled the distance f(x) = (x/10) + 100 yard.

This function as a model of the movement (or rather the relative movement of the tortoise to the "standing" position of Achilles) now enables us to reproduce the paradox on a new level because of its double character: The continuous aspect of the movement does not contradict the discrete perspective. It is the representation in terms of the function concept that, first of all, enables us to free Achilles' movement from the one-sided fixation on the series of distinct points x_n (n = 0, 1 ....), and also to see the movement as a whole.

The relative movement of Achilles and the tortoise is a linear function, as both movements are uniform: f(x) = ax +

b (i.e., when Achilles reaches x, the tortoise is at f(x)). The problem: “At what point does Achilles really catch up with the tortoise?” is now: “What is the fixed point of f(x)?” The fixed point of f(x) can be calculated simply as a function of the constants a and b: x = f(x) = ax + b. We seemingly have solved the problem by taking a relational point of view, which means, by adopting a “world view” which provides objects and relations between objects with an equal ontological status.

This essentially makes up for what has been called a transition from thinking about objects to a complementary relational thinking. This transition took place at the end of the 18^{th} century only. In what sense is this a solution? The paradox of the movement leads to a complementarity in the concept of “function”!

However, this complementarity requires a genetic and dynamical view on mathematics, not as a fixed logical edifice, but as an evolving and growing body of knowledge and activity.

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