DEVELOPING A FRAMEWORK AND THE CONSTRUCTION OF AN UNDERSTANDING OF PLACE VALUE

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ABSTRACT

In an effort to make sense of prospective teacher's views of decimal fractions, an exploration of facets of place value understanding was conducted. To move past a focus on performance as a measure of understanding, a framework for the exploration of place value ideas as they relate to prospective teachers' understanding of decimal fractions was developed. The framework is used to analyze the work of a small collection of prospective elementary teachers who worked to share their understanding of decimals. Conclusions regarding the power of the inquiry in mathematics teacher educator's understandings of learners and as a springboard for instruction are shared.

Keywords: Place value, Decimals, Prospective Elementary Teachers.

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The historical development of the Hindu Arabic numeration system illustrates the struggle of human beings to develop notational systems that are both efficient and illustrative. Perhaps the key to the efficiency of the Hindu Arabic numeration system is the characteristic of place value. The mathematical definition of place value fails to illuminate struggles human beings face in constructing numerals to represent quantities of different sizes. Researchers have illustrated the significant struggles of children as they make sense of numeration (Kamii, 1985; Steffe, Cobb, & von Glasersfeld, 1988) and difficulties that can remain after high school (Putt, 1995; Thipkong & Davis, 1991). Research exploring prospective and inservice teachers' understandings of place value have illuminated the challenge professionals face as they work to make sense of and use numeration (McClain, 2003; Thanheiser, 2009; Widjaja, Stacey, & Steinle, 2008). These studies inform our understanding of the reasoning used by prospective teachers (teachers). Our understanding of our students' reasoning requires that we build models of our students' mathematics (Steffe, 1990) to guide our teaching. These models have the potential to inform our teaching in ways that a focus on errors cannot. As we engaged with teachers we began to think about nuances of the approaches we took with them in light of the way they approached the tasks we gave them. We worked to explore our students' reasoning to begin to build our understandings of our practice as mathematics teacher educators. In the course of this investigation we progressed from studying our students to understanding our view of our students as sources for the development of our own knowledge.

The development of content knowledge, including understandings of commonly used notations and methods or algorithms, is one of several goals in prospective elementary teacher education programs (Fennema & Franke, 1992; Putnam, Heaton, Prawat, & Remillard, 1992). Yet it was our own love of mathematics and a desire to understand that mathematics as it was represented in the marks and utterances of our students that motivated us to look more deeply into the understandings of our students. Our investigation of teachers' understandings of decimal fractions (decimals) and ultimately place value began as we noticed, after 9 semester hours of mathematics content course work, students in our elementary teacher education certification program had difficulty with decimals. As we watched our teachers in the field and read their papers, ideas about their understandings and links to their analysis of children's work began to form. We could hear and feel the challenges the teachers faced and felt as we read their narratives and watched them engage with children about decimals. Their efforts to make sense of the children's mathematics (Steffe, 1990) seemed constrained by their views of the decimals themselves. As educators of these prospective teachers, we felt the weight of our responsibility to identify "the problem" and "fix" it. This was our initial orientation to the challenge of teaching our students. Over the following sections we share our journey to understand our students, identify facets of our work as teacher educators, and our learning resulting from our findings. Our work contributes to the growing body of literature that illustrates the challenges of mathematics teacher educators as they work to build practices that will support the development of useful understandings of mathematics for mathematics teaching and learning. Ultimately our specific goal was to make sense of the thinking of our students and to develop learning opportunities for them that build from existing understandings of decimals in particular, and place value in general.

UNDERSTANDING OUR PROBLEM

Our work began as we explored the literature on teacher's understandings of decimal fractions and found the work of Putt (1995). The task he presented seemed quite straightforward and potentially revealing (see Figure 1). At this point in our investigation we were still voyeurs of performance. We peered at percentages of correct and incorrect answers, still awed by the overwhelming notion of lost opportunities for learning the percentages represented. Initially, not really believing the percentages could apply to our students in our context, we presented an adapted version of the task to our own students. The adaptation was drawn from our own experiences and from Putt's discussion of student responses that suggested some students might see zero as larger than any decimal. We added zero to the numerals in Putt's ordering task and gave the task to our students.

Place the following numbers in order from smallest to largest.

0.606, 0.0666, 0.6, 0, 0.66, 0.060

Illustrate and explain your thinking about this problem.

Figure 1: Decimal Ordering Task

Our Students

We used the decimal ordering task in three of our classes during the spring 2004. Author 1 taught a mathematics course for 22 prospective elementary school teachers preparing to enroll in a certification program. The six-semester hour course met three times per week for 15 weeks. The decimal ordering task was given prior to instruction in number and operation, but following two weeks of activity focused on the development of problem solving and metacognitive skills. Author 2 taught two sections of a mathematics methods course. The course was the second of two mathematics methods courses in the four semester long program. In the first semester our students had studied pre-kindergarten to second grade students' learning of mathematics. The focus was the development of curiosity about mathematics and the mathematics of children. Building an awareness of one's own understandings of mathematics and of children was a hallmark of the course. Students engaged with children in school contexts. In the second semester course, teaching and learning of upper elementary children (ages 8 – 12) was studied with a continued focus on understanding mathematics and the mathematics of children. The three-semester hour course met for four hours once each week for 15 weeks. Fifty-one students in two sections of the course completed the task during the sixth class meeting, prior to field work. All students in Author 2's classes had experience in their first semester course working one-on-one with a child to explore mathematical understandings.

In all three classes students worked independently and finished the task in 20 minutes. In the methods courses the students were asked to reflect on and revisit the task for homework. We were hopeful that this opportunity would afford students the opportunity to explore and expound on their ideas. We also provided the students three square grids. Each grid contained a square partitioned into ten, one hundred, and one thousand parts (we call these tenths, hundredths, and thousandths grids) respectively. We were unsure how the students might use the grids, but were hopeful that they would encourage explorations of the values of the numerals involved in the decimal ordering task. The students submitted their written work at the next class session.

As soon as the students shared their work with us, we rushed to search for errors and compared the different orderings generated to the findings of Putt's. We found no obvious differences in the mathematics and mathematics methods students' performance. Our tallies of correct and incorrect answers seemed dissatisfying as the lens of performance voyeur did not help us understand how to interact with our students. Putt's (1995) careful attention to students' written comments encouraged us to dig more deeply into the student work samples and to explore existing research to gain insight regarding teacher's reasoning about decimals.

Making Sense of Teachers' Work on Decimal Tasks

Beginning with Grossman's (1983) report of performance on university entrance examination we found examples of the challenge of interpreting and using decimals. Grossman noted that students asked to compare decimals found the task difficult. This finding was consistent with a robust collection of studies of children's work on decimal comparison tasks (Resnick et al., 1989; Sackur-Grisvald & Leonard, 1985). These studies categorized and identified reasons for learners' errors tied to an understanding of place value and the complexity of decimal notation. Resnick et al. (1989) noted that decimal notation masks the denominator of the fractions being represented. We also considered the flexibility in thinking the decimal notation affords, allowing for infinitely many interpretations of the decimals in relation to various powers of ten. In our exploration of the literature we found what at first appeared to be conflicting evidence regarding teacher performance on decimal comparison tasks (Stacey et al., 2001) and Putt's (1995) ordering task. Further exploration revealed the work of Sackur-Grisvald and Leonard (1985), that described tasks involving multiple comparisons as generating increased errors among children (ages 11-14) who "made almost no errors when comparing two decimals" (p. 167).

So it seemed plausible that Putt's task would elicit a substantial number of errors from adults.

Of the 704 teachers in Putt's study, 360 ordered the decimals correctly. Putt reported that of the 256 teachers who selected 0.6 as the largest decimal, 198 created one of two orderings {0.0666, 0.060, 0.606, 0.66, 0.6; or 0.0666, 0.606, 0.060, 0.66, 0.6}. According to Putt, teachers generating these orderings appeared to be using either the *zero* or *fraction rule*. The zero rule, as described by Sackur-Grisvald and Leonard (1985), involves identifying decimals with zero tenths as smaller than those with a non-zero digit in the tenths position. The fraction rule involves reasoning that the shorter the decimal the more tenths it contains and hence the larger the quantity it represents. Conjectures regarding the teachers' reasoning were based on written explanations given and interviews. Responses given by teachers who ordered the decimals in one of the two ways identified generally did not rely on taught rules, but explained understandings abstracted from their experiences with decimal fractions and common fractions. For example, "If there is a 0 in front of it and it's longer, more numbers in it, then it is smaller. When you have a fraction with the big number on the bottom it's smaller, it's the same sort of thing" (Putt, 1995, p. 8). These teachers reasoned using their understanding of place value and the unit associated with the position of the last digit in the numeral.

Consistent with the hypothesis of Sackur-Grisvald and Leonard (1985) prospective teachers are more successful on decimal comparison tasks. Stacey et al. (2001) gave 553 prospective or inservice teachers attending four different universities a 27 question decimal comparison test. Stacey et al. (2001) found that most of the teachers could correctly compare decimals, but had more difficulty when they were asked to compare a decimal between zero and one to zero. Stacey suggested that the teachers were reasoning about zero as if it were a whole number, again pointing to understandings of place value as the source of the teachers' sense making.

Twenty-eight of our 73 prospective elementary teachers put the numerals in correct order. The remaining students made errors identified in the literature by several researchers (Hiebert, 1985; Irwin, 2001; Khoury & Zazkis, 1994; Nesher & Peled, 1986; Putt, 1995; Stacey, et al., 2001). Eleven of the students put the numbers in the correct order but incorrectly placed 0 as the largest number. Ten of these student explanations were consistent with those identified by Putt (1995) and Stacey et al. (2001). The remaining students did not explain why zero was listed as having the greatest value. Preliminary exploration of the other 34 responses revealed that some students used rules, identified in the literature (Sackur-Grisvald & Leonard, 1985, Resnick et al., 1989) to order the decimals. Two students ordered the numbers using the *whole number rule*, more digits indicates a larger value. Twenty-three used the fraction rule. Other students worked to minimize the complexity of the task or reduce the number of decimal places. For example, some students decided to make the number of decimal places equal in all the numerals and then were unsure how to proceed. Others multiplied all the numerals by a number that would "result in whole numbers" but in picking the multiplier they chose a number like 5 and found a new collection of decimals to order.

As constructivist teachers (Steffe & D'Ambrosio, 1995) we felt the discomfort of a living contradiction (Whitehead, 1989). While we profess to believe in the primacy of a teacher's attention to the sense making activity of learners, our first attempt to explore our own students' thinking was by attending not to sense making, but to performance. Reading literature on decimal performance that included hypotheses about possible reasoning (Putt, 1995; Resnick, et al., 1989) encouraged us to think deeply about how the students might be reasoning in their efforts to complete the decimal ordering task. We identified students' understanding of place value as a source of reasoning about decimals. At this point we put the student work samples aside for several months and began developing what became our description of an understanding of place value. As we developed a description we considered existing literature that described place value (Fuson, 1990; Hart, 1989; McClain, 2003) and considered evidence we had seen in our interactions with students. We focused our discussions on hypothetical mental objects and actions. Several times we disagreed on elements of what became our framework, however we came to agreement by returning to examples drawn from our experience with children and adults.

PLACE VALUE AS A SYSTEM OF UNITS

Our discussion of place value will be situated in the Hindu-Arabic numeration system, however the ideas apply equally well to numeration systems in other bases (Khoury & Zazkis, 1994; Zazkis & Khoury, 1993). The mathematical definition of place value explains very little about how it can be understood. We provide a definition focused on how learners construct meaning for and represent quantity. This definition draws on the idea that the construction of units and relationships between them are the mental objects used to make sense of and use a numeration system.

We define place value as *a system of units and relationships among those units*. The development of this system begins with the identification of a unit and the development of larger and smaller units using two related actions. First, ten copies are made of the unit (generative unit). Using this multiplicative action a collection of units is generated. This set of ten units, can also be thought of as a unit of ten. The multiplicative action can then be applied to each of the ten units or to the unit of ten, to generate ten sets of ten units or a unit of one hundred. Thus a recursive process comprised of repeated applications of the multiplicative action and initialization of the unit is used to build units larger than the generative unit. The second action used to create units is division. The generative unit is divided into ten equal pieces. Each of the pieces can then be considered a new unit, one tenth, to which the division action can again be applied (see Figure 2).

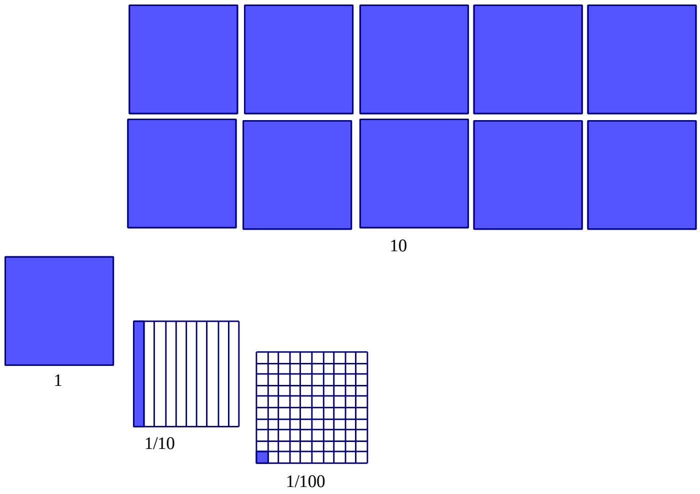
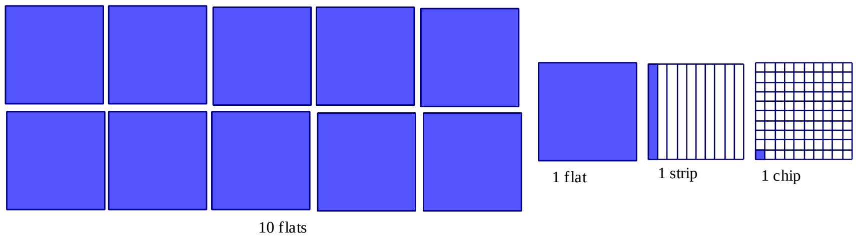


Figure 2: Results of Iterative Process on a Unit

As with the multiplicative action, the division action is part of an iterative process that can be used to generate a desired unit. Thus a collection of units associated with a generative unit {…, 100, 10, 1, 1/10, 1/100, …} is constructed. This collection of units in sequence can form a sequence of slots (Steffe, 1994). The generative unit, 1, can thus be any number, and the remaining terms in the sequence, also units, are associated with multiples or parts of the generative unit. For example, if the generative unit is 245, then the sequence of terms, also units would be {… , 24,500, 2,450, 245, 24.5, 2.45, …}.

Because the sequence of units is built using the actions of multiplication and division, relationships between the units can be identified. Two relationships exist between any pair of units. First, one unit is strictly greater than the other. This ordering relation is rather weak, but establishes a way of organizing and reasoning about the units. Thus given the units 10 and 1/100, ten is greater than one hundredth. Second, each unit can be considered as a part of the other. Making sense of this may necessitate the coordination of a pair of unit sequences. For example, consider the sequence of units illustrated in flats in Figure 3. Ten flats can be thought of as 1 unit. Based on this initialization, 1 flat is one tenth of the unit, 1/10 of the flat is one hundredth of the unit, and 1/100 of the flat is one-thousandth of the unit.



sequence A	1 unit	1/10	1/100	1/1000
sequence B	10	1	1/10	1/100
sequence C	100	10	1	1/10
sequence D	1000	100	10	1

Figure 3: Association of Sequence of Units with Various Quantities.

As is illustrated in Figure 3 the value of the unit depends on the situation at hand. For example, a collection of ten objects can be thought of as 1 unit (see sequence A in Figure 3). Using this assignment, 1 flat is 1/10 of the unit and 1 strip is 1/100 of the unit. Sequences B, C, and D illustrate the relationships among the quantities of flats and parts of flats when the unit is identified as 1 flat, 1 strip, and 1 chip respectively. Thus if 1 chip is thought of as a unit (see sequence D in Figure 3), then 10 flats contain 1000 chips.

For a robust understanding of place value, the learner must be able to think of a quantity as a multiple of units and as a sum of multiples of different units. This thinking known as the additive characteristic of the numeration system is traditionally captured in what is taught as expanded notation. Most often students view this approach to the decomposition of a number as a procedure. The procedure is fairly simple to replicate: for example 452 = 400 + 50 + 2. One need only know the units associated with each position and write the number as a sum of multiples of the different units. This seemingly trivial procedure that is easily mimicked for whole numbers, is essential in a robust understanding of place value. Consider two decimals: 0.66 and 0.606. These decimals can be written as fractions: $\frac{66}{100}$ and $\frac{606}{1000}$. Because these two fractions have unlike denominators, to compare the two it is not uncommon for students to write the first as $\frac{660}{1000}$ prior to making a comparison. This procedure is unnecessary for the learner with a robust understanding of place value. He or she need only decompose the numbers 0.66 and 0.606 into 0.6 + 0.06 and 0.6 + 0.006 respectively and use his or her understanding of the relative size of the units, 0.01 and 0.001 to order the decimals. The use of the additive characteristic of the numeration system allows the learner to compose or decompose a number from multiples of units and to select an appropriate construction in many different circumstances.

Our interpretation of place value as a system of units and relationships is a lens through which to view our students' actions and explanations. We provide one caveat before continuing. Although we have described several elements of a robust understanding of place value, we are not suggesting that these elements develop in isolation or in a linear fashion. The development of one unit, for example understanding that a collection of 10 objects can also be thought of as one group of 10, does not imply the development of all units. Thus the framework does not suggest to us levels of expertise, but rather a means for making sense of the evidence of the mathematics of adult learners (Steffe, 1988). Evidence from research on children's understanding of number suggests that the development of a system of units and relationships is anything but linear (Kamii, 1986; Steffe, 1988).

The framework provided a way for us to make sense of the work samples through our identification of evidence of the use of units, relationships, and the additive characteristic. At this point in our investigation we turned back to the work samples intent on using the framework to make sense of our students' reasoning. We hoped that what at first might have appeared to be very similar responses, on closer examination using the elements of the framework could reveal complex and different understandings of decimal notations and place value. Any differences we identified, we reasoned, would be of use in helping us develop instruction and productive ways of interacting with our students. We also felt that applying the framework to the work samples would help us evaluate its utility. Including whether and how it helped us build understanding of our students' reasoning and whether there were elements missing from the framework or not useful in building models of our students' reasoning.

ANALYSIS OF UNDERSTANDING AND THE POWER OF THE FRAMEWORK

To focus our thinking on the possibilities in our students' reasoning, we initially explored work samples of students who correctly ordered the decimals. These students shared reasoning steaming from the use of algorithms and procedures, but there were also explanations that revealed more robust understandings. These responses encouraged us to look carefully at all the responses of the methods students' homework papers. We found ten work samples that coupled with the initial ordering task, contained powerful evidence of understanding of place value when viewed through the framework. We discussed the language and notations used in these responses and made conjectures about possible meanings in light of the framework. In particular we looked for evidence of units, relationships, and the additive characteristic. Differences in language and notations that became relevant in our discussions are shared in our descriptions of the explanations given by students in the next section. While we realize the components of the framework for place value understanding may be related or linked, focusing our lens on one component at a time during the analysis allowed us to begin to build models of student's understandings. We remind the reader that when we use the term models we are describing our view of the student's mathematics (Steffe, 1990) built to inform our teaching. From this perspective the models are effective only if they are useful to us in our practice as teachers.

OUR MODELS OF UNDERSTANDING GENERATED WITH THE FRAMEWORK

In the following sections we describe the types of explanations given by students and make interpretations that serve as models of our understanding of our students' reasoning.

Explanations Based on Learned Algorithms

For some students learned procedures or algorithms combined, in some cases, with understandings of some units allowed them to order the decimals correctly. In addition, the explanations they provided allowed us to make sense of the approaches that enabled them to successfully complete the tasks. Generally speaking students using algorithms or procedures applied digit comparison or transformed the numbers in ways that allowed them to use their whole number reasoning.

The responses labeled as a, b, c all apply the digit by digit comparison method to generate the order.

a. a method similar to alphabetizing words

> *"Zero is the smallest number because the only numbers smaller than zero are negative numbers. Then I looked at the tenths place to see which number comes next. 0.060 and 0.0666 both have zeros in the tenths place so I looked at the hundredths place. These are also the same, both sixes, so I looked at the thousandths place to see that .060 has a zero in the place, which is smaller than the six in the other numbers so I knew .060 was the next smallest numbers, then .0666 was next. Now I have two numbers left, .6 and .606. .6 is smaller because it doesn't have any smaller number like .606 does in the hundredths or thousandths place."*

In response a, the student seems to apply the algorithm with expert precision focusing on digits in places. The student refers to the digit in the tenths place, rather than using the language of tenths. The response caused us wondered about the 0.6 and 0.606 comparison and whether this was evidence of the additive characteristic. We concluded that questioning about such a response during class time could help us explore the student's use of additivity to reason about the comparison.

b. grouping of numbers that have the same digit in the tenths place

> *"I knew that the #s [numbers] .6, .606, .66 would be the highest grouping because each has 6 tenths compared to the others with 0 tenths. As each # had 6 in the tenths place I looked to the next spot (hundredths) so I saw that .66 was the next highest because there were 6 hundredths. Between .6 and .606 I had to look at the thousandths spot because both numbers had 0 hundredths. .606 had more thousandths so I know it was highest. I then repeated this with*

> *.060 and .0666, noticing that .0666 was higher because it had 6 thousandths where .060 had none."*

We interpreted response b as different from response a in two ways. First the student shares that she is thinking of tenths, rather than a digit in the tenths position. Second she sorts the decimals into two groups based on the quantity of tenths represented in the decimal notation. This representation of decimals as quantities of units, suggested to us that this student had an understanding of the decimal representation of units underlying her seemingly algorithmic explanation. We saw evidence of an understanding of the additive nature of the decimal notation, one of the three components of our framework.

c. equalizing the number of digits to the right of the decimal point

> *"Since 0 represents the quantity of nothing it is the smallest. I added 0's to the other numbers to the ten-thousandths place value so that all place value columns could be compared for quantity. I looked at the tenth place value column of each and saw that 0.66 has 6 tenths, so does 0.606. Comparing the hundredth's place value columns, 0 is less than 6, so 0.606 < 0.66. I continued to compare each place value column to order the numbers as shown."*

Response c is different from response b in that the student first appends zeros to fill the positions to the ten-thousandths place. This student did not compare the numbers as whole numbers. Instead, the student refers to the 6 in the "tenths place value column" as a quantity of tenths. The remaining "place value columns" involve digit comparison "0 is less than 6." For us this meant that the student may have a unit of a tenth, but chose to resort to an analysis digit by digit to order the decimals beyond the initial comparison of the number of tenths.

In the last three responses of this type, students' transform the given numerals to perform the task.

d. moving the decimal point to the right two places, comparing the resulting percents

> *"At first I just tried to line them up at a glance. But then as I double checked them, I thought of them as percents and had to do some re-arranging. I moved the decimal 2 places to the right to make them 0%, 6.0%, 6.6%, 60.6%, and 66%."*

This student is willing to be transparent about her confusion regarding how to order the decimals and her work to reduce her confusion. She cleverly transformed the decimals to percents using a learned procedure. This enabled her to compare whole numbers of percents to order the decimals. We view this student's approach as an algorithm because she needed to change the notational domain to one with fewer digits after the decimal to make sense of quantities that could be compared. We suspect that the size of units, thousandths and ten-thousandths, and relationships among units was a source of confusion for the student. Notation that included tens and hundreds was more meaningful than notation involving thousandths and ten-thousandths.

e. Changing decimal numbers to fractions, creating common denominators and comparing the numerators as whole numbers

> *"If you make them fractions and all their denominators are 10,000 (the largest), then you discover which totals more.*

$.606 = \frac{606}{1{,}000} = \frac{6060}{10{,}000}$

$.0666 = \frac{666}{10{,}000} = \frac{666}{10{,}000}$

$.6 = \frac{6}{10} = \frac{6{,}000}{10{,}000}$

0 =

$.66 = \frac{66}{100} = \frac{6{,}600}{10{,}000}$

$.060 = \frac{6}{100} = \frac{600}{10{,}000}$ *"*

Response e also illustrates that the student shifted to a different notational system to make sense. Shifting all of the decimals to equivalents with a common denominator can be interpreted in many ways, but we suggest two plausible explanations here. First, the student has developed units, in this case a unit of one ten-thousandth, but may not yet have the additive characteristic (for example, 0.606 is not 0.6 + 0.006). Without the additive characteristic the student uses a single unit, ten-thousandths, to make sense of each decimal quantity. Second, and for us more plausible is that for the student the decimals are not notations that make sense. Converting to fraction notation allows her to make sense of the quantities as multiples of a common unit. In this case the student can then make the comparison using whole numbers.

f. *Moving the decimal points four places in order to make all the numbers whole numbers*

> *"The easiest way to figure is to move the decimal places so as to make the numbers look like whole numbers. Example: you have to move all of them 4 places, so:*

.606	=	*6060*	*5*
.0666	=	*666*	*3*
.6	=	*6000*	*4*
0	=	*0*	*1*
.66	=	*6600*	*6*
.060	=	*600*	*2"*

Response f was identified by Resnick et al. (1989) in their analysis of understanding used by children when they employ the appending zeros procedure and then could easily compare decimals using whole number reasoning. The student articulates clearly that making the comparison is easy if you "make the numbers look like whole numbers."

These students' responses helped us understand the challenge they faced as they worked to make sense of the decimal notation. To face the challenge the students seemed to work to use primarily whole number thinking to make sense of the task. In order to use such reasoning, the students showed ability to transform decimal quantities into multiples of a unit or whole numbers. Students also used understandings of units such as tenths, to create groups of decimals to order and then finished the task by comparing digits. These methods enabled them to complete the task and us to understand our challenge as teachers to create opportunities for the construction of units and the use of such constructions to make comparisons between decimals.

Gaining More Insight from Explanations

We now turn to student work that we identified as illustrating a more robust understanding of place value. The work of students who articulated their thinking in either the initial task or the subsequent homework around units, provided evidence of understanding the relationships among units, and considered the additive nature of the system of units to have a robust understanding of place value.

In the first work sample the student seems to recognize the additive nature of the decimal place value system. He breaks each number into decimal parts and highlights the additive nature of the representation. We are unsure whether this student is considering the relationships between the different decimal parts. For example, does the student see hundredths as tenths of tenths? Does he see the relationship between the places in the decimal number? Does this student consider a unit as a referent for the decimal parts?

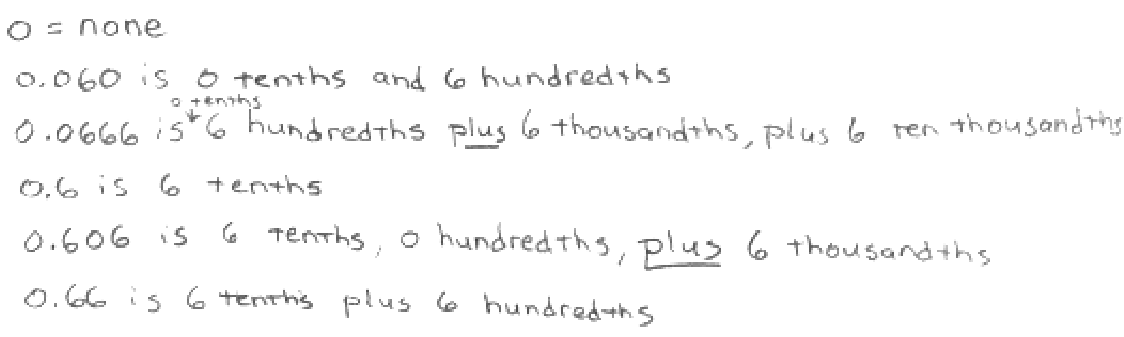


Figure 4: Dan's Explanation of the Decimal Ordering Task

Some students used the grids provided by the teacher to illustrate their thinking about the comparison of the different numbers. It is conceivable that a student could explain the decimal representations as Dan has (see Figure 4) while struggling to produce a pictorial representation. The pictures students used seem to indicate attention to a particular unit that serves as a referent for the use of decimal representations. Illustrations like Gary's (see Figure 5) also often reveal emergent understanding of relationships between the different decimal places.

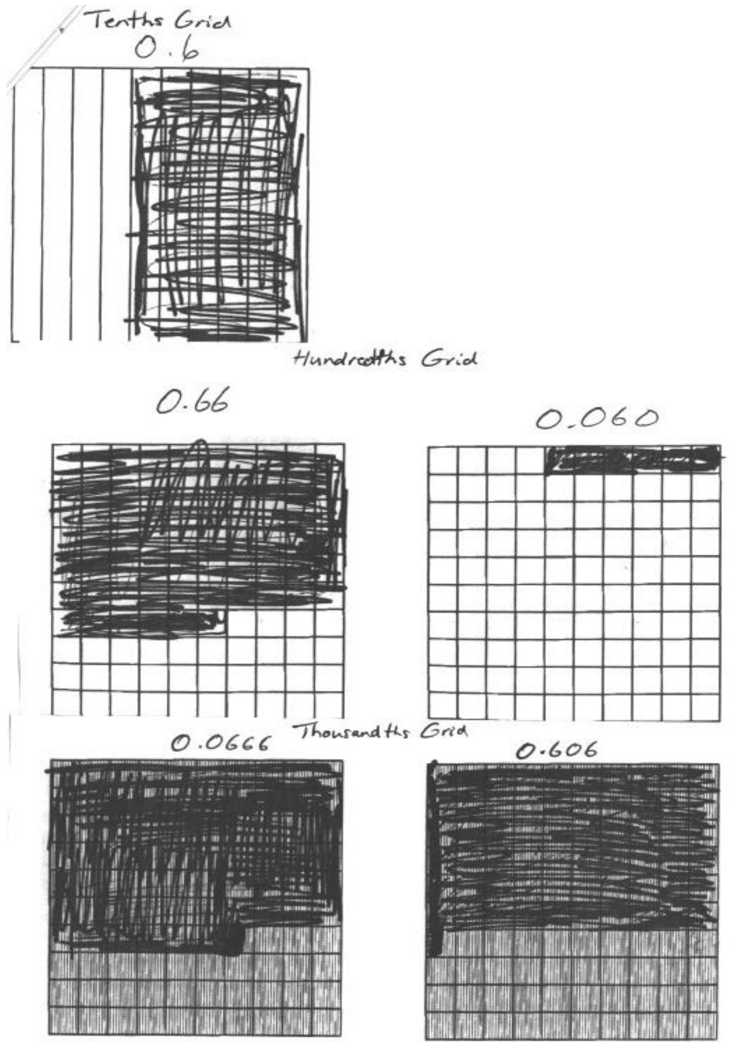


Figure 5: Gary's Explanation of His Decisions for the Decimal Ordering Task

Gary described his approach, "So the strategy I used was to shade in the values of each number on grids handed out in class. In doing so I saw which decimal number was smallest and largest depending on how much of the grid was shaded." Gary represented tenths on what he called a tenths grid, the hundredths on the hundredths grid (broken into 100 parts) and the thousandths on the grid broken into

1000 parts. It is not clear whether he made sense of the grids as representing a common unit. The number 0.0666 appears to challenge Gary's strategy. One plausible explanation for his approach lies in the fact that there was no grid broken into ten-thousandths. Since he relied on certain grids to represent certain decimal values we contend that he had not yet considered relationships between the recurrent subdivisions and was unable to extrapolate from the picture of thousandths to the subdivisions necessary to represent ten-thousandths. Further evidence supporting this explanation is drawn from his initial response to the decimal ordering task based on the fraction rule {0, 0.0666, 0.606, 0.060, 0.66, 0.6}. This reasoning seems to stem from a comparison of parts rather than a consideration of quantity of parts. We share an excerpt from Gary's initial explanation to illustrate his thinking: "Though 0.0666 has more numerals, it has a smaller value because it has the 10 thousanths [sic] place a smaller percentage of a whole." We interpret this approach as evidence of an ordering relation on the units 0.0001, 0.001, 0.01, and 0.1. In moving to represent the numbers on the grids, Gary only faces difficulty when a grid representing a multiple of the units he needs to illustrate is not given.

Nadia grapples with this same problem of how to subdivide beyond the thousandths grid, but seems to be a step beyond in her thinking. She explains:

> *"Using the grid paper begin by shading 0 blocks because the smallest number has no numbers. Next you would shade 60 blocks in a thousand (sic thousandths) grid. Then you would move up by .0666 by shading 60 blocks and another 6 blocks plus 1/6 of a block. For .6 you would shade 600 blocks [referring to each of the subdivisions in the thousandths grid] and then add 6 more blocks for .606. Last would be 600 and 60 blocks [referring to .66]."*

There are two important components of understanding revealed in this student's work. Her consistent use of the thousandth grid for all the representations suggests that she is attending to her need for the definition of a unit. Furthermore, Nadia, in contrast to Gary, is able to consistently use the thousandth grid for all the representations, suggesting that she comprehends how many thousandths make one hundredth or one tenth, yet she struggles to move from a thousandth to a ten-thousandth. The relationship between the different decimal parts seems tenuous. Her subdivisions result in shading 1/6 of one-thousandth in order to represent 6 ten-thousandths. Despite the obvious practical challenge of partitioning thousandths on the given grids into ten-thousandths, Nadia, like Gary, appears to struggle only when she attempts to construct a unit less than thousandths. She effectively interprets 0.060 as 60 “blocks” and 0.6 as 600 “blocks” evidence that she may have constructed relationships between the selected part, thousandths, and larger parts (e. g. tenths). The additive nature of the composition of the decimal representation of values is also apparent in Nadia’s work, as was the case in Dan’s work. She demonstrates that 0.606 is a composite unit of 600 blocks, or thousandths, and 6 more blocks.

Yet another student, Paula, describes her thinking using the same thousandths grid (see Figure 6). Paula’s work contains clearer evidence than Nadia that a relationship has been built between thousandths and other parts of the units such as tenths. First, Paula refers to the parts of the unit as fractions (e. g. 6/10) rather than “blocks.” Second, the marks she uses to indicate 0.6 of the unit (see number 6 in Figure 6) suggest she can see one-tenth in a thousandths grid. This practice and her use of 2 by 3 rectangular regions to indicate six-hundredth may have been used to represent the 6/10000 of a unit. The region Paula shaded to represent 6/10000 appears to be proportional to the 2 by 3 rectangular region she shaded to represent 6/100. This suggests that Paula may have shaded six-hundredths of one-hundredth to represent 6/10000. Still, given the lack of clarity in the drawing, it is difficult to be sure of Paula’s approach. Paula’s work also reveals the focus on the additive nature of the components of the decimal numbers similar to Dan’s.

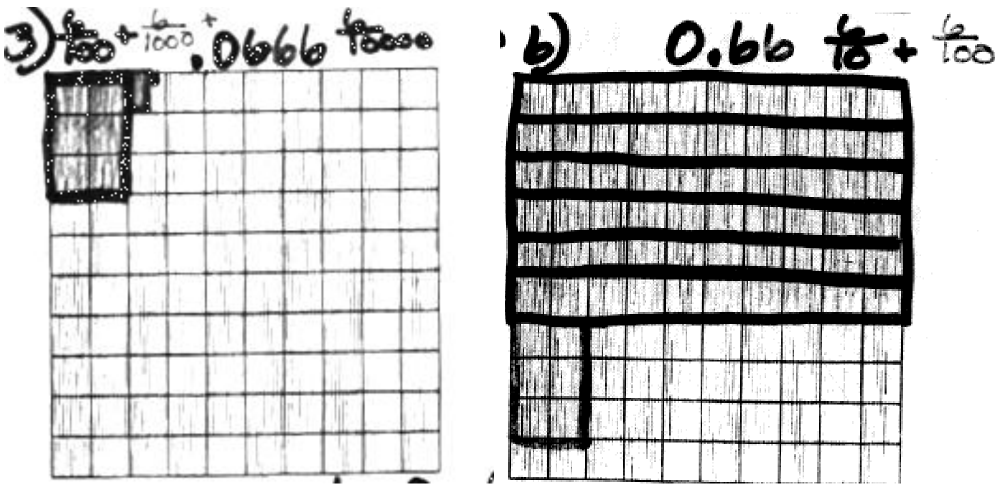


Figure 6: Paula’s Explanations for the Decimal Ordering Task

The next work sample corroborates the findings from the work of the previous students. Samantha is clearly using a unit that is broken successively into tenths in order to produce the many different components of the decimal numbers in the task (see Figure 7). The sample demonstrates the student’s attempt to draw a representation that illustrates her more formal reasoning. The reasoning, represented symbolically, suggests that she interprets a string of decimal parts additively. From her use of fractions we might infer that she has an awareness of the relationships between the parts. In other words, that she perceives hundredths as tenths of tenths. Her picture does not help us feel totally confident of this component of her reasoning, since she does not extend her drawings beyond hundredths and appears to use two different partitionings of one to illustrate hundredths and tenths. Instead she uses a picture of a unit broken into tenths with six tenths shaded and the same size unit broken into hundredths with six hundredths shaded in order to compare six-hundredths to six-tenths.

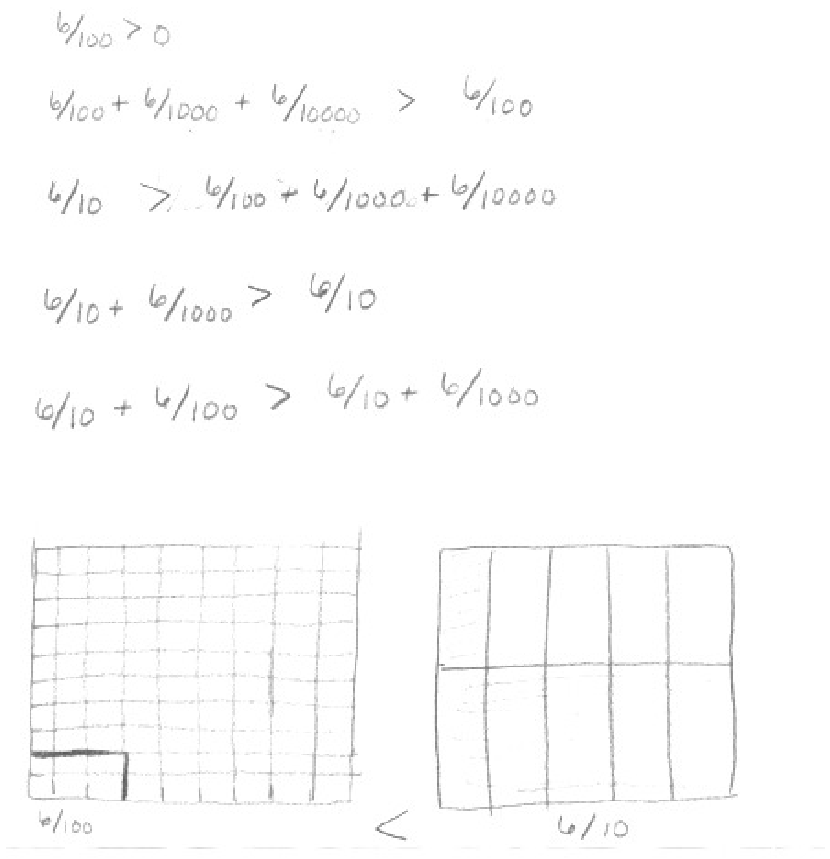


Figure 7: Samantha’s Explanations for the Decimal Ordering Task

In the next work samples the students associate the task to a context. Two students explained that they made sense of numbers involving decimals by thinking about money.

> *"The first two spots to the right of the decimal can be used in money relation (cents of a dollar), anything after that is a fraction of a penny."*

This student was not only using a familiar context to help her make sense of the decimals, but she was articulate in defining the unit that was to serve as the referent for her thinking (i.e. the dollar) and the need to think of the penny as another unit, thus allowing her to further subdivide the dollar. She exemplifies her understanding of these multiple units when she describes .0666. "I look at it as 6¢ and 2/3 of another penny = 6 2/3 ¢. …. (.606) is slightly bigger than 60¢ because 6/10 of a penny is 3/5. Causing (.606) = 60 3/5 ¢." In referring to .66 she considered it "closest to a dollar." This student provides evidence of three important components of her sense making about decimals. First, she is able to use more than one unit as needed in thinking about the values of the numbers and the various decimal places. Second, she uses the additive nature of the numeration system to make sense of how quantities are composed. So in thinking of the decimal places beyond the first two (which define cents) she understands that parts of that smallest unit, a cent, accumulate and are added to however many cents are represented by the number. So decimal places beyond the hundredths suggest that more value is accumulating. Finally, she has a strong sense of the relationship between the parts of units (dollars or cents).

Two other students provide evidence of using a context to make sense of the ordering numbers task. In one case the student suggests she converts all of the decimals to percents and then reasons about the value of the percents in order to place the numbers in order. Her example is the following: "if someone has 100 pieces of candy and our neighbor gets 66% and you get 0% -- clearly they got the better deal with 66 pcs [pieces] vs [versus] your 0 pieces." Although it is unclear whether this student is flexible in thinking about units different than 100, she is clearly articulating the need for reference to a unit, in this case, 100. The second student suggests, but does not provide much detail, that she thinks about measurement, whenever she approaches a task of this sort. She also drew a number line to assist in her reasoning. Although her work is very vague, "I tried to imagine measuring something," it does provide some evidence of the need for a decision regarding a unit with which to work, in particular the (0,1) interval on the number line.

Finally, Stella's work sample illustrates flexibility in thinking about the relationships between the parts (see Figure 8). In this work we notice that Stella creates a picture to show the subdivisions. It is not troubling to her that she must magnify the part in order to clearly show continued subdivisions. This shift suggests that she can think of two units simultaneously. She takes one-tenth, this becomes a new unit that is broken into tenths again. She is able to rename the resulting pieces as hundredths of the original whole, yet her picture shows tenths. She repeats this process, demonstrating flexibility in her use of units. The emphasis of her reasoning lies in the relationships among the different places and the partition used to create the parts of a unit they represent. Furthermore, she seems to rely on the additive nature of the place value system.

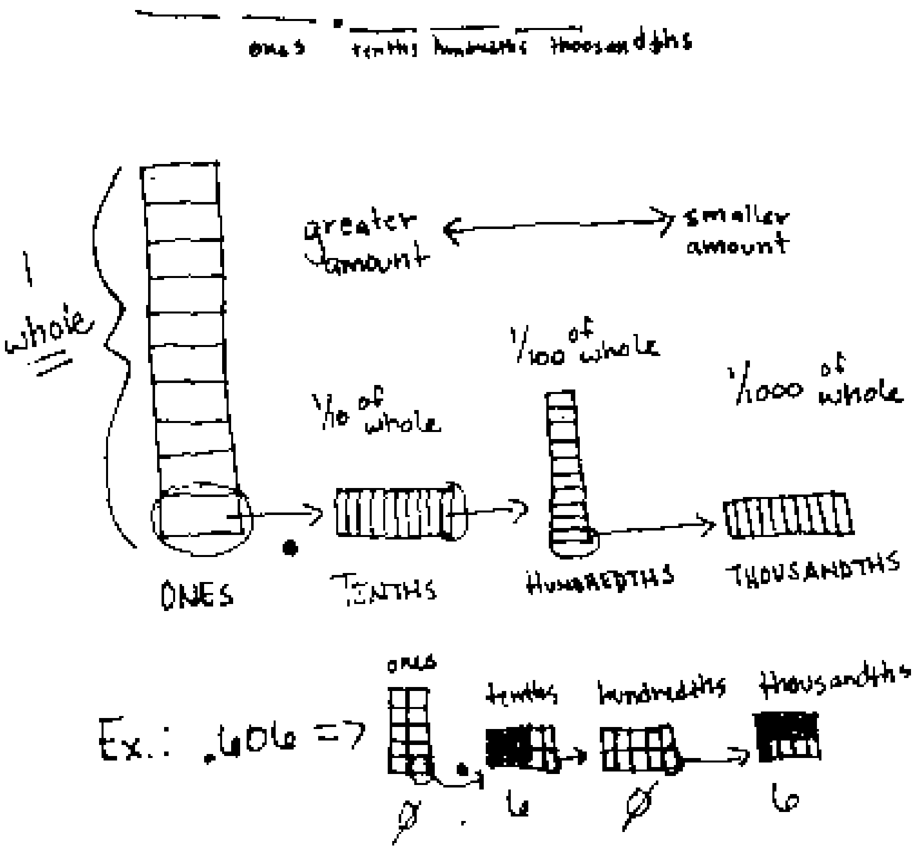


Figure 8: Stella’s Explanation for the Decimal Ordering Task

Although Stella’s thinking seems very similar to that of the student who used money to describe her decision-making for ordering numbers involving decimals, her work seems to have been generalized. Her thinking is not bound to a context. It is conceivable that students who rely on a context for this task may struggle to dissociate the work from the particular context. We are unsure how these students would behave in a task that was bound to a context different than money, percents of quantities of candy, or measurement. The power in being able to generate a context in which the numerals mean for the student illustrates a way of operating that could be useful in instruction.

These student work samples provide evidence that they were thinking of units and their parts (with or without context), that there was understanding of the relationships between the units and the parts, and that they saw the place value system as additive in nature. These facets of the framework were very useful to us in making sense of our students' reasoning and beginning to see decimal quantities, as our students, did. These students have thought about ordering decimals in ways that go beyond the more typical use of a school-learned procedure.

CONCLUSIONS: LOOKING BACK AND LOOKING FORWARD

Our journey in this exploration has included several transitions. We began with a quest to understand the reasoning of our students when we were initially surprised at our students' struggles with decimals. As constructivist teachers (Steffe & D'Ambrosio, 1995) we knew that models of students' understanding should serve as the basis for our choices for teaching. The focus on errors found in the literature was not useful in our efforts to build models on which we could base our teaching. We turned to the development of an epistemic subject (Sierpinska, 1994) for whom we theorized a robust understanding of place value. This work resulted in a framework for our analysis of students' understanding of place value. Use of the framework enabled us to build models, grounded in student work, which allowed us to make sense of the complexity of developing a robust understanding of place value.

The explanations of students revealing a more robust understanding of place value illustrate some of the challenges faced by mathematics educators as they attempt to design opportunities for the development of this understanding. The students whose work we shared, not only ordered the decimals, but provided explanations that allowed us to gain insight into their reasoning. We identified two sorts of responses from these students: explanations that contained evidence of a more robust understanding of place value and explanations that were algorithmic. Using the framework to explore the explanations of students with a more robust understanding of place value revealed that even when students correctly order decimal fractions, their understanding of place value may still be developing. The work of Nadia and Paula serves as evidence that students may struggle with relationships between units. The complexity of this ordering task as indicated by Sackur-Grisvald and Leonard (1985) may have challenged the students' thinking and revealed evidence of existing difficulties with relationships between less often used units. Our models suggests that while these students may have constructed a relationship between the adjacent units of tenths and hundredths, relationships may not yet have been constructed between other less familiar adjacent place-value units such as thousandths and ten-thousandths.

The work of students relying on context to order decimals was at first glance reminiscent of the approaches identified by Stacey and Steinle (1998). Some students in Stacey and Steinle's work had difficulty when they were asked to compare decimal fractions that included units less than one hundredth. In our work, the students who used context to make sense were also able to transform the decimal fractions into whole numbers and fractions of cents and percents. Their approaches viewed through the lens of our framework suggest that the students using context are working with the development of related values based on a common unit. What is not clear using the framework is whether the students' approach is drawn from a structural understanding of place value or if the decimals were tied to the context used by the students. Using very different means of communicating, Stella, Samantha, and Dan's work samples each contain evidence of units, relationships, and additivity. Perhaps with the exception of Stella, these three student's representation leave open to question the structure of their understanding of units and relationships. Our interactions with students in the context of teaching would focus on trying to gain further insight into students' use of units, relationships, and additivity.

The majority of our students who correctly ordered decimals provided evidence of application of algorithms and whole number reasoning. This finding encouraged us to reflect on our practice. The algorithmic (rule based) explanations that we provoked had been included in previous analyses of student work and included appending zeros, writing the decimals as fractions with common denominators, and "alphabetizing" (Putt, 1995; Resnick, et al., 1989; Stacey & Steinle, 1998). While we had expected such responses from students in the mathematics content class, we were concerned when after a semester long methods course during which students were encouraged to look for, verbalize, and build conceptual understanding the students still turned to algorithms and rules to justify their answers. It appears to us that our efforts to support sense making regarding decimals had failed for these students.

As we reflected on this failure, we recognized that we were traversing familiar ground. Almost constantly, we assume that our students "understand" concepts that are viewed in as "elementary." With no careful examination of the experiences of our students or consideration of exploring their understanding we work to teach concepts that in part rely on understandings that our students have had no or limited opportunity to develop. The lessons we should have learned by reading about the experiences of our colleagues (Confrey, 1991), seem far from our day to day engagement with students. The living contradiction that motivated a more detailed look at our students' work allowed us to see the effects of our faulty assumptions about our students.

We both found that the framework and the individual models developed from student work were critical to our development as mathematics teacher educators. The framework allowed us to reason in a general way about student understanding and to unpack important elements of student understanding of decimals. It also gave us greater insights into the incompleteness of many adults' understanding of place value for whole numbers. In turn, the application of the framework to student work samples allowed us to understand the individual manifestations of understanding in new ways. It allowed us to give reason to our students and to begin to understand the nature of the differences in their thinking and strategies. While the framework points to large landmarks on the canvas of our understanding, the individual models provide a closer more careful view of the terrain. We can understand the difference between the challenge of building the relationships between one and a tenth and the very different challenge of building a relationship between a tenth and a thousandth (or any other non-adjacent units). We are more aware of the efforts of students to make sense of the additive nature of the place-value system and our need to provide tasks that will foster a greater depth of understanding of that particular attribute of the system. We are more purposeful in questioning students' solutions based on the three components of the framework, units, relationships and additivity. Most importantly, we are more aware of how difficult it is to acquire a robust understanding of place value and how many of our students rely on learned algorithms in order to successfully complete place value tasks. Our tasks and our discussion lack constraints that would encourage the development of the elements identified in the framework, such as units and relationships, still be developing.

Our challenge as mathematics teacher educators is to design tasks that encourage students to build new ways of operating and to share these understandings with us. Building from the whole number and algorithmic approaches we identified, toward the construction of tenths, for example, might be as easy as asking the students to represent the quantity of tenths in two square cakes. Those students who reasoned with context afforded us the opportunity to see the power in their approach and to identify ways to build from contexts toward more general structural reasonings of place value.

Our framework was useful in helping us understand where we were falling short as teachers and beginning to identify areas of challenge our students faced. The models we developed became levers in our own thinking, used to interact with students around the concept of place value as it is used by the students as they represent and reason about decimal fractions. As we move forward in our work with students the framework we view the framework as a tool to support further studies of prospective teachers' understanding of place value and the development of these types of tasks that encourage place value understanding.

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